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A Class of New Hybrid Algebraic Multilevel Preconditioning Methods*

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ABSTRACT

A class of new hybrid algebraic multilevel preconditioning methods is presented for solving the large sparse systems of linear equations with symmetric positive definite coefficient matrices resulting from the discretization of many second-order elliptic boundary-value problems by the finite-element method. The new preconditioners are shown to be of optimal orders of complexities for two-dimensional and three-dimensional problem domains, and their relative condition numbers are estimated to be bounded uniformly, independent of the numbers of both the levels and the nodes. © Elsevier Science Inc., 1997

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1. INTRODUCTION

For the large sparse systems of linear equations with symmetric positive definite (SPD) coefficient matrices generated by the discretization of many second-order self-adjoint elliptic boundary-value problems through the finite-element method, the algebraic multilevel preconditioning methods proposed by Axelsson and Vassilevski (see [1]) in 1989 are the most effective ones for numerical solution. Following the ways of constructing the preconditioners of Axelsson and Vassilevski, nowadays, many researchers such as Axelsson, Vassilevski, and Evans (see [2–3, 5–12]) have studied extensively and developed in depth both the preconditioner designs and the theoretical analyses of the methods in terms of algebraic multilevel preconditioning.

In this paper, based on the already existing results mentioned above, we establish a class of new algebraic multilevel preconditioning methods for solving large sparse systems of linear equations. These novel preconditioners are substantially technical combinations and reasonable generalizations of the above-stated ones, and are shown to be much more effective and applicable. In a careful way, we estimate the relative condition numbers corresponding to the new preconditioners and calculate the computational work loads for the resulting preconditioning methods. It is concluded that our new preconditioners are of optimal order of complexity for two-dimensional (2D) and three-dimensional (3D) problem domains, and their relative condition numbers are not only independent of the regularity of the solution, but also bounded uniformly with respect to the levels and with respect to the possible jumps of the coefficients of the considered second-order self-adjoint elliptic boundary-value problem as long as they occur only across edges (faces in 3D) of elements from the coarsest triangulation.

This work can be thought as an extension of [1–3, 7, 9] and an improvement of [6, 8, 10–11] too.

2. CONSTRUCTION OF THE PRECONDITIONERS

We carry on the algebraic treatment for the second-order self-adjoint elliptic boundary-value problem (see [1]). Let $\{A^{(k)}\}_{k=1}^{l}$ and $\{\bar{A}^{(k)}\}_{k=1}^{l}$ be the stiffness matrices according to the well-formed two-level hierarchical bases and the nodal bases of a given sequence of finite-element spaces. Then they naturally admit the following two-by-two block forms:

$$A^{(k)} = \begin{pmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A^{(k-1)} \end{pmatrix} \in L(R^{n_k}), \qquad A^{(k-1)} \in L(R^{n_{k-1}}), \quad (2.1a)$$

$$\bar{A}^{(k)} = \begin{pmatrix} \bar{A}_{11}^{(k)} & \bar{A}_{12}^{(k)} \\ \bar{A}_{21}^{(k)} & \bar{A}_{22}^{(k)} \end{pmatrix} \in L(R^{n_k}), \qquad \bar{A}_{22}^{(k)} \in L(R^{n_{k-1}}), \tag{2.1b}$$

where n_k denotes the number of the nodes on level k. Moreover, there exists a sequence of block upper triangular two-by-two matrices

$$J^{(k)} = \begin{pmatrix} I & J_{12}^{(k)} \\ 0 & I \end{pmatrix}$$

such that

$$A^{(k)} = J^{(k)T}\bar{A}^{(k)}J^{(k)T}. \tag{2.2}$$

We use $S^{(k)}$ and $\bar{S}^{(k)}$ to denote the Schur complements of $A^{(k)}$ and $\bar{A}^{(k)}$, respectively, i.e.,

$$S^{(k)} = A^{(k-1)} - A_{21}^{(k)}A_{11}^{(k)^{-1}}A_{12}^{(k)},$$

$$\bar{S}^{(k)} = \bar{A}_{22}^{(k)} - \bar{A}_{21}^{(k)}\bar{A}_{11}^{(k)^{-1}}\bar{A}_{12}^{(k)} \equiv S^{(k)}.$$

For the requirements of designing the new preconditioners, we assume:

ASSUMPTION (A_1). $B_{11}^{(k)}$ ($k = 1, 2, \ldots, l$) are given SPD matrices such that

$$v_1^T A_{11}^{(k)} v_1 \leqslant v_1^T B_{11}^{(k)} v_1 \leqslant (1 + b_k) v_1^T A_{11}^{(k)} v_1, \qquad k = 1, 2, \ldots, l$$

hold for all $v_1 \in R^{n_k - n_{k-1}}$ and some $b_k \geqslant 0$ ($k = 1, 2, \ldots, l$).

ASSUMPTION (A_2). $p_\nu(t)$ ($0 \leqslant t \leqslant 1$) is a given nonincreasing polynomial of degree ν such that

$$p_\nu(0) = 1, \qquad 0 \leqslant p_\nu(t) < 1 \quad (0 < t \leqslant 1).$$

If we write

$$Q_{\nu-1}(t) = \frac{1 - p_\nu(t)}{t}, \tag{2.3}$$

then $Q_{\nu-1}(t)$ is clearly a polynomial of degree $\nu - 1$ and satisfies

$$Q_{\nu-1}(t) > 0 \quad (0 < t \leqslant 1), \qquad Q_{\nu-1}(0) = -p_\nu'(0). \tag{2.4}$$

Practically, $p_\nu(t)$ can be taken to be the properly scaled and shifted Chebyshev polynomial

$$p_\nu(t) = \frac{T_\nu\left(\dfrac{1+\alpha-2t}{1-\alpha}\right)+1}{T_\nu\left(\dfrac{1+\alpha}{1-\alpha}\right)+1}, \qquad 0 < \alpha < 1, \tag{2.5}$$

or the polynomial

$$p_\nu(t) = (1-t)^\nu. \tag{2.6}$$

Here, T_ν denotes the νth-degree Chebyshev polynomial. About the properties of these two kinds of polynomials, one can see [1–2] for details.

It is easy to see that if $\alpha = 1$, (2.5) turns to (2.6) and that both the polynomials $p_\nu(t)$ defined in (2.5) and (2.6) satisfy Assumption (A_2), while the one given by (2.5) has the smallest local maximum in the interval $[\alpha, 1]$ $(0 < \alpha < 1)$. Moreover, there hold

$$p_\nu(\alpha) = \frac{2}{T_\nu\left(\dfrac{1+\alpha}{1-\alpha}\right)+1}, \qquad p_\nu(1) = \frac{(-1)^\nu+1}{T_\nu\left(\dfrac{1+\alpha}{1-\alpha}\right)+1}.$$

With the above preparations, the preconditioners can be readily constructed based on approximation matrices of the stiffness matrices at the given discretization levels. After factorizing each of these approximation matrices into block triangular factors, we replace (approximate) the resulting Schur complement by the approximation matrix corresponding to the preceding (coarser) level. This process is repeated successively for a fixed number k_0 $(1 \leqslant k_0 < l$, a given integer) of steps. After each k_0 steps, the preconditioner so derived is corrected by a certain polynomial approximation which involves the preconditioner and the approximation matrix at the considered level. Evidently, these new ideas are significantly different from the existing ones for designing the preconditioners just from the original stiffness matrices rather than approximations of those matrices.

The approximation-matrix sequence can be formed in various ways. Here we only consider two typical cases. One recursively defines the current approximation matrix by applying the approximation matrix with respect to the preceding level thoroughly, while the other recursively defines it in a similar way, but a restarting strategy is used after each k_0 steps.

The afore-described methods can now be formulated mathematically as follows:

First, we introduce the auxiliary matrix sequence $\{B^{(k)}\}$ based on $\{B_{11}^{(k)}\}$ and $\{A^{(k)}\}$ in accordance with either of the following two methods:

Method (I). $B^{(1)} = A^{(1)}$, and $\{B^{(k)}\}_{k=2}^{l}$ is recursively defined by

$$B^{(k)} = \begin{pmatrix} B_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & B^{(k-1)} \end{pmatrix}, \qquad k = 2, 3, \dots, l.$$

Method (II). $B^{(1)} = A^{(1)}$, and $\{B^{(k)}\}_{k=2}^{l}$ is recursively defined by

$$B^{(k)} = \begin{pmatrix} B_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & \tilde{B}^{(k-1)} \end{pmatrix}, \qquad k = 2, 3, \dots, l,$$

with

$$\tilde{B}^{(k-1)} = \begin{cases} A^{(k-1)} & \text{if } \quad k = sk_0, \\ B^{(k-1)} & \text{otherwise}, \end{cases} \qquad k = 2, 3, \dots, l, \quad s = 1, 2, \dots, l(k_0),$$

where $l(k_0) = l/k_0 - 1$.

Then, by making use of these two kinds of matrix sequences, we define the new hybrid algebraic multilevel preconditioners $\{M^{(k)}\}$ for the two-level hierarchical basis stiffness matrices $\{A^{(k)}\}$ paralleling to those in [10] as follows:

$$M^{(1)} = A^{(1)},$$

$$M^{(k)} = \begin{pmatrix} B_{11}^{(k)} & 0 \\ A_{21}^{(k)} & \tilde{M}^{(k-1)} \end{pmatrix} \begin{pmatrix} I & B_{11}^{(k)-1} A_{12}^{(k)} \\ 0 & I \end{pmatrix}, \qquad k = 2, 3, \dots, l, \tag{2.7}$$

where for the fixed positive integer k_0,

$$\tilde{M}^{(k-1)} = \begin{cases} \hat{M}^{(k-1)} & \text{if} \quad k = sk_0 + 1, \\ M^{(k-1)} & \text{otherwise}, \end{cases}$$

$$k = 2, 3, \ldots, l, \quad s = 1, 2, \ldots, l(k_0), \tag{2.8}$$

while

$$\hat{M}^{(k-1)} = \begin{cases} \tilde{S}^{(k)}\left[I - p_\nu\left(M^{(k-1)^{-1}}\tilde{S}^{(k)}\right)\right]^{-1}, & \text{version (i)}, \\ B^{(k-1)}\left[I - p_\nu\left(M^{(k-1)^{-1}}B^{(k-1)}\right)\right]^{-1}, & \text{version (ii)}, \end{cases} \tag{2.9}$$

with

$$\tilde{S}^{(k)} = B^{(k-1)} - A_{21}^{(k)} B_{11}^{(k)^{-1}} A_{12}^{(k)} \tag{2.10}$$

being either the Schur complement of $B^{(k)}$ according to method (I) or the approximated Schur complement of $B^{(k)}$ according to method (II).

Evidently, when we take $B_{11}^{(k)} = A_{11}^{(k)}$ ($k = 2, 3, \ldots, l$) for method (I) or $k_0 = 1$, $B_{11}^{(k)} = A_{11}^{(k)}$ ($k = 2, 3, \ldots, l$) for method (II), these methods reduce to those of Axelsson and Vassilevski [1], and when we take $B_{11}^{(sk_0)} = A_{11}^{(sk_0)}$ [$s = 1, 2, \ldots, l(k_0)$], Method (II) gives the method discussed in Vassilevski [10]. However, it is noted that the methods in Axelsson and Vassilevski [2] are not included in these new methods. On the other hand, by reasonably choosing the polynomials in the above methods, we can obtain a series of new algebraic multilevel preconditioning methods. Thus, our new methods are reasonably general.

Since the new preconditioners are constructed starting from an approximation-matrix sequence for the stiffness-matrix sequence rather than starting from the stiffness-matrix sequence itself and then approximating $A_{11}^{(k)}$ by $B_{11}^{(k)}$ ($k = 2, 3, \ldots, l$), we only need to form the whole stiffness matrices at the finest and the coarsest levels in the implementations of the new methods. In addition, compared with the methods in [2, 10], the new methods only need the calculations of $B^{(k-1)}v$ ($k = 2, 3, \ldots, l$) instead of $A^{(k-1)}v$ ($k = 2, 3, \ldots, l$) for some $v \in R^{n_{k-1}}$ ($k = 2, 3, \ldots, l$). Therefore, if $\{B^{(k)}\}$ is chosen to have a good sparse property, the computational costs of our new methods can then be considerably decreased.

Moreover, our new methods also remove the strict requirement that the approximation matrices $B_{11}^{(k)}$ ($k = 2, 3, \ldots, l$) be spectrally equivalent to the matrices $A_{11}^{(k)}$ ($k = 2, 3, \ldots, l$), respectively, in a uniform manner for all k [see Assumption (A_1)], which is indispensable in the related existing results.

The above are all advantages of our new hybrid algebraic multilevel preconditioning methods over the known ones.

Alternatively, corresponding to another auxiliary matrix sequence $\{\overline{B}^{(k)}\}$ defined in accordance with $\{B_{11}^{(k)}\}$ and $\{\overline{A}^{(k)}\}$ by

$$\overline{B}^{(k)} = J^{(k)-T} B^{(k)} J^{(k)-1},$$

we can also analogously construct another class of new hybrid algebraic multilevel preconditioners $\{\overline{M}^{(k)}\}$ for the nodal basis stiffness matrices $\{\overline{A}^{(k)}\}$ as

$$\overline{M}^{(k)} = J^{(k)-T} M^{(k)} J^{(k)-1}.$$

By direct computations, we can immediately obtain the concrete expressions of $\overline{B}^{(k)}$ and $\overline{M}^{(k)}$.

3. PRELIMINARY ANALYSIS

In this section, we are going to make some essential preparations for estimating the relative condition numbers of the two-level hierarchical basis stiffness matrices with respect to their corresponding preconditioners defined in last section.

LEMMA 1 (See [10]). *Let*

$$v = \left(v_1^T, v_2^T\right)^T \in R^{n_{k+k_0}}, \qquad v_2 \in R^{n_k}.$$

Then

$$v_2^T A^{(k)} v_2 \leqslant \eta(k_0) v^T A^{(k+k_0)} v. \tag{3.1}$$

The function $\eta = \eta(k_0)$ ($k_0 \geqslant 0$) is an increasing function of k_0 independent of k. More precisely, the following asymptotic behavior holds:

$$\eta(k_0) = \begin{cases} Ck_0 & \text{for a } 2D \text{ domain}, \\ C\mu^{k_0} & \text{for a } 3D \text{ domain}. \end{cases} \tag{3.2}$$

The constant $\mu \geqslant 2$ is an upper bound of the ratio of the mesh sizes h_k and h_{k+1} of two consecutive grids, that is, $\mu \geqslant \max_{1 \leqslant k \leqslant l-1} h_k/h_{k+1}$. The constant C is independent of possible jumps of the coefficients of the considered elliptic boundary-value problem as long as they are discontinuous only across edges (faces in $3D$) of elements from the initial triangulation.

LEMMA 2 (See [4]): (A strengthened Cauchy-Bunyakowskii-Schwarz (CBS) inequality). *There holds*

$$\left|v_1^T A_{12}^{(k)} v_2\right| \leqslant \gamma\left(v_1^T A_{11}^{(k)} v_1\right)^{1/2}\left(v_2^T A^{(k-1)} v_2\right)^{1/2}$$

for all $v_1 \in R^{n_k - n_{k-1}}$ *and* $v_2 \in R^{n_{k-1}}$, *where* $\gamma = \sqrt{1 - 1/\eta(1)} < 1$, *uniformly in* $k = 2, 3, \ldots, l$.

LEMMA 3 (See [4]). *There holds*

$$v_1^T A_{11}^{(k)} v_1 \leqslant \eta(1) v^T A^{(k)} v \qquad \forall v = \left(v_1^T, v_2^T\right)^T \in R^{n_k}, v_2 \in R^{n_{k-1}}.$$

LEMMA 4 (See [4]). *There holds*

$$v_2^T A_{21}^{(k)} A_{11}^{(k)-1} A_{12}^{(k)} v_2 \leqslant \gamma^2 v_2^T A^{(k-1)} v_2 \qquad \forall v_2 \in R^{n_{k-1}}.$$

LEMMA 5 (See [7]). *There holds*

$$(1 - \gamma^2) v_2^T A^{(k-1)} v_2 \leqslant v_2^T S^{(k)} v_2 \leqslant v_2^T A^{(k-1)} v_2 \qquad \forall v_2 \in R^{n_{k-1}}.$$

LEMMA 6. *Let Assumption* (A_1) *be satisfied. Then there holds for* $\forall v_2 \in R^{n_{k-1}}$

$$v_2^T S^{(k)} v_2 \leqslant v_2^T \hat{S}^{(k)} v_2 \leqslant \frac{v_2^T S^{(k)} v_2 + b_k v_2^T A^{(k-1)} v_2}{1 + b_k},$$

where

$$\hat{S}^{(k)} = A^{(k-1)} - A_{21}^{(k)} B_{11}^{(k)-1} A_{12}^{(k)}.$$

Proof. Similar to the proof of Lemma 6 in [8], so it is omitted. ∎

LEMMA 7. *Let Assumption* (A_1) *be satisfied with*

$$b_k \leqslant b_0 q^{l-k}, \qquad k = 1, 2, \ldots, l \tag{3.3}$$

for some $b_0 > 0$ *and* $q \in (0, 1)$. *Then* $\forall k \in \{1, 2, \ldots, k_0\}$, $\forall s \in \{0, 1, \ldots, l(k_0)\}$, *there holds*

$$1 \leqslant \frac{v^{(sk_0+k)T} B^{(sk_0+k)} v^{(sk_0+k)}}{v^{(sk_0+k)T} A^{(sk_0+k)} v^{(sk_0+k)}} \leqslant 1 + \eta_{s+1}(k_0),$$

where

$$\eta_{s+1}(k_0) = \begin{cases} q^{l-k_0}\phi(k_0)\eta(k_0)^s \sum_{j=0}^{s} \left[q^{k_0}\eta(k_0)\right]^{-j} := \eta_{s+1}^{(\mathrm{I})}(k_0), \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{method } (\mathrm{I}), \\ q^{l-(s+1)k_0}\phi(k_0) \qquad\qquad\qquad := \eta_{s+1}^{(\mathrm{II})}(k_0), \\ \qquad\qquad\qquad\qquad\qquad\qquad \textit{method } (\mathrm{II}), \end{cases}$$

$$\phi(k_0) := b_0\eta(1) \sum_{i=0}^{k_0} q^i\eta(i),$$

and it is stipulated from now on that $\eta(0) \equiv 1$.

Moreover, $\eta_s(k_0)$ $(s = 1, 2, \dots, l/k_0)$ *are increasing functions about both* k_0 *and* s *for both method* (I) *and method* (II), *and can be bounded uniformly with respect to* $s = 1, 2, \dots, l/k_0$ *from above by*

$$\bar{\eta}(k_0) = \begin{cases} \psi(k_0) := \dfrac{\phi(k_0)}{1 - q^{k_0}\eta(k_0)}, & \textit{method } (\mathrm{I}), \\ \phi(k_0), & \textit{method } (\mathrm{II}), \end{cases}$$

provided $q^{k_0}\eta(k_0) < 1$ *for method* (I).

Proof. Take

$$v^{(k)} = \left(v_1^{(k)T}, v_2^{(k)T}\right)^T \in R^{n_k}, \qquad v_2^{(k)} \in R^{n_{k-1}}, \qquad v^{(k-1)} = v_2^{(k)}.$$

Since for $\{B^{(k)}\}$ defined by method (I) there holds

$$v^{(k)T}\left(B^{(k)} - A^{(k)}\right)v^{(k)} = v_1^{(k)T}\left(B_{11}^{(k)} - A_{11}^{(k)}\right)v_1^{(k)} + v^{(k-1)T}\left(B^{(k-1)} - A^{(k-1)}\right)v^{(k-1)}, \quad (3.4)$$

while for $\{B^{(k)}\}$ defined by method (II) there holds

$$v^{(k)T}\left(B^{(k)} - A^{(k)}\right)v^{(k)} = v_1^{(k)T}\left(B_{11}^{(k)} - A_{11}^{(k)}\right)v_1^{(k)} + v^{(k-1)T}\left(\tilde{B}^{(k-1)} - A^{(k-1)}\right)v^{(k-1)}, \quad (3.5)$$

considering Assumption (A_1) and making use of induction, we can immediately get that

$$v^{(k)T}(B^{(k)} - A^{(k)})v^{(k)} \geqslant 0, \qquad k = 1, 2, \dots, \tag{3.6}$$

holds for the sequence $\{B^{(k)}\}$ defined by either method (I) or method (II). Hence, the left-hand side of the inequality sought is valid.

Now, $\forall k \in \{sk_0 + 1, sk_0 + 2, \dots, (s+1)k_0\}$, by using (3.4)–(3.5) recursively the following inequality can be derived:

$$\begin{aligned} v^{(k)T}(B^{(k)} - A^{(k)})v^{(k)} &\leqslant v^{(k-j)T}(B^{(k-j)} - A^{(k-j)})v^{(k-j)} \\ &\quad + \sum_{i=k-j+1}^{k} v_{\mathrm{I}}^{(i)T}(B_{\mathrm{II}}^{(i)} - A_{\mathrm{II}}^{(i)})v_{\mathrm{I}}^{(i)} \\ &\quad j \in \{1, 2, \dots, k - sk_0\}, \quad s = 0, 1, \dots, l(k_0). \end{aligned}$$

In particular, $\forall k \in \{1, 2, \dots, k_0\}$ and $\forall s \in \{0, 1, \dots, l(k_0)\}$, there holds

$$\begin{aligned} &v^{(sk_0+k)T}(B^{(sk_0+k)} - A^{(sk_0+k)})v^{(sk_0+k)} \\ &\leqslant v^{(sk_0)T}(B^{(sk_0)} - A^{(sk_0)})v^{(sk_0)} + \sum_{j=sk_0+1}^{sk_0+k} v_{\mathrm{I}}^{(j)T}(B_{\mathrm{II}}^{(j)} - A_{\mathrm{II}}^{(j)})v_{\mathrm{I}}^{(j)}. \end{aligned} \tag{3.7}$$

By Lemma 3 and Assumption (A_1) as well as Lemma 1, we have

$$\begin{aligned} &\sum_{j=sk_0+1}^{sk_0+k} v_{\mathrm{I}}^{(j)T}(B_{\mathrm{II}}^{(j)} - A_{\mathrm{II}}^{(j)})v_{\mathrm{I}}^{(j)} \\ &\qquad \leqslant \sum_{j=sk_0+1}^{sk_0+k} b_j v_{\mathrm{I}}^{(j)T} A_{\mathrm{II}}^{(j)} v_{\mathrm{I}}^{(j)} \\ &\qquad \leqslant \sum_{j=sk_0+1}^{sk_0+k} b_j \eta(1) v^{(j)T} A^{(j)} v^{(j)} \\ &\qquad = \sum_{i=1}^{k} b_{sk_0+i} \eta(1) v^{(sk_0+i)T} A^{(sk_0+i)} v^{(sk_0+i)} \\ &\qquad \leqslant \sum_{i=1}^{k} b_{sk_0+i} \eta(1) \eta(k-i) v^{(sk_0+k)T} A^{(sk_0+k)} v^{(sk_0+k)} \\ &\qquad \leqslant \sum_{i=0}^{k-1} b_0 q^{l-(sk_0+k-i)} \eta(1) \eta(i) v^{(sk_0+k)T} A^{(sk_0+k)} v^{(sk_0+k)}, \end{aligned}$$

where in the last inequality we have used the condition (3.3). Substituting this estimate into (3.7) we obtain that

$$
\begin{aligned}
& v^{(sk_0+k)T}\left(B^{(sk_0+k)}-A^{(sk_0+k)}\right)v^{(sk_0+k)} \\
& \quad \leqslant v^{(sk_0)T}\left(B^{(sk_0)}-A^{(sk_0)}\right)v^{(sk_0)} \\
& \qquad + b_0\eta(1)q^{l-sk_0-k}\sum_{i=0}^{k-1} q^i\eta(i)v^{(sk_0+k)T}A^{(sk_0+k)}v^{(sk_0+k)} \qquad (3.8)
\end{aligned}
$$

hold for all $k=1,2,\dots,k_0$, $s=0,1,\dots,l(k_0)$.

Based on (3.8), we can assert that

$$
\begin{aligned}
& v^{(sk_0+k)T}\left(B^{(sk_0+k)}-A^{(sk_0+k)}\right)v^{(sk_0+k)} \\
& \quad \leqslant \bar{\eta}_{s+1}(k_0)v^{(sk_0+k)T}A^{(sk_0+k)}v^{(sk_0+k)}, \\
& k=1,2,\dots,k_0, \quad s=0,1,\dots,l(k_0), \qquad (3.9)
\end{aligned}
$$

hold with

$$
\bar{\eta}_1(k_0)=q^{l-k_0}\phi(k_0),
$$

$$
\bar{\eta}_{s+1}(k_0)=\begin{cases} q^{l-(s+1)k_0}\phi(k_0)+\eta(k_0)\bar{\eta}_s(k_0), & \text{method (I)}, \\ q^{l-(s+1)k_0}\phi(k_0), & \text{method (II)}, \end{cases}
$$

$$
s=1,2,\dots,l(k_0). \qquad (3.10)
$$

In fact, noticing the expression for method (II), from (3.8) it follows that

$$
\begin{aligned}
& v^{(sk_0+k)T}\left(B^{(sk_0+k)}-A^{(sk_0+k)}\right)v^{(sk_0+k)} \\
& \quad \leqslant v^{(sk_0-1)T}\left(\tilde{B}^{(sk_0-1)}-A^{(sk_0-1)}\right)v^{(sk_0-1)} \\
& \qquad + q^{l-(s+1)k_0}\phi(k_0)v^{(sk_0+k)T}A^{(sk_0+k)}v^{(sk_0+k)}. \qquad (3.11)
\end{aligned}
$$

We see easily that (3.9) is true for method (II).

Furthermore, by applying induction and making use of Lemma 1, we know from (3.8) that

$$\begin{aligned}
&v^{(sk_0+k)T}\big(B^{(sk_0+k)} - A^{(sk_0+k)}\big)v^{(sk_0+k)}\\
&\quad\leqslant \overline{\eta}_s(k_0)v^{(sk_0)T}A^{(sk_0)}v^{(sk_0)} + q^{l-(s+1)k_0}\phi(k_0)v^{(sk_0+k)T}A^{(sk_0+k)}v^{(sk_0+k)}\\
&\quad\leqslant \big[\overline{\eta}_s(k_0)\eta(k) + q^{l-(s+1)k_0}\phi(k_0)\big]v^{(sk_0+k)T}A^{(sk_0+k)}v^{(sk_0+k)}\\
&\quad\leqslant \overline{\eta}_{s+1}(k_0)v^{(sk_0+k)T}A^{(sk_0+k)}v^{(sk_0+k)}.
\end{aligned}$$

Therefore, (3.9) is also true for method (I). Now, applying (3.10) corresponding to method (I) regressively, we can obtain

$$\begin{aligned}
\overline{\eta}_{s+1}(k_0) &\leqslant \eta(k_0)^s\overline{\eta}_1(k_0) + \sum_{i=2}^{s+1}\eta(k_0)^{s+1-i}q^{l-ik_0}\phi(k_0)\\
&= \eta(k_0)^s\overline{\eta}_1(k_0) + \eta(k_0)^{s+1}\sum_{i=2}^{s+1}\big[\eta(k_0)^i q^{(i-1)k_0}\big]^{-1}\overline{\eta}_1(k_0)\\
&= q^{l-k_0}\phi(k_0)\eta(k_0)^s\sum_{i=0}^{s}\big[\eta(k_0)q^{k_0}\big]^{-i} = \eta_{s+1}^{(\mathrm{I})}(k_0),
\end{aligned}$$

which together with (3.9)–(3.10) readily implies the validity of the first conclusion of Lemma 7.

The remaining conclusions of Lemma 7 can now be obtained by direct calculations, and there is no need for us to demonstrate them in detail. ∎

REMARK 3.1. The condition (3.3) can be satisfied by constructing the matrices $B_{11}^{(k)}$ ($k = 1,2,\dots,l$) in a way similar to [2]. Concretely speaking, for $k = 1,2,\dots,l$ we let SPD matrices $G_{11}^{(k)}$ be the incomplete triangular factorizations of the matrices $A_{11}^{(k)}$ such that $G_{11}^{(k)}$ are convergent splittings of $A_{11}^{(k)}$, respectively, i.e., $\rho(I - G_{11}^{(k)^{-1}}A_{11}^{(k)}) \leqslant \tilde{q} < 1$, and the nonzero elements in each row of the matrices $G_{11}^{(k)}$ have the same order $[O(1)]$ as those of the matrices $A_{11}^{(k)}$ with their number being fixed, where we use $\rho(\cdot)$ to denote the spectral radius of the corresponding matrix. Define $C_{11}^{(k)} = I - G_{11}^{(k)-1}A_{11}^{(k)}$ and $B_{11}^{(k)} = A_{11}^{(k)}[I - C_{11}^{(k)2\beta_k}]^{-1}$ with $\beta_k = m(l-k+1)$ ($m \geqslant 1$ an integer) for $k = 1,2,\dots,l$. Then it is easily seen that

$$v_1^TA_{11}^{(k)}v_1 \leqslant v_1^TB_{11}^{(k)}v_1 \leqslant \frac{1}{1-\tilde{q}^{2\beta_k}}v_1^TA_{11}^{(k)}v_1, \qquad k = 1,2,\dots,l.$$

So Assumption (A_1) is valid with

$$b_k = \frac{\tilde{q}^{2\beta_k}}{1-\tilde{q}^{2\beta_k}} = \frac{\tilde{q}^{2m}}{1-(\tilde{q}^{2m})^{l-k+1}}(\tilde{q}^{2m})^{l-k}.$$

If $\tilde{q}^{2m} < 1-\gamma^2$, then (3.3) is satisfied with $b_0 = q/(1-q)$ and $q = \tilde{q}^{2m}$.

REMARK 3.2. Asymptotically, there holds

$$\max\{\phi(k_0), \psi(k_0)\} = O(1),$$

provided $q < 1$ for a 2D problem domain and $q < \mu^{-1}$ for a 3D problem domain, where μ is the same as in Lemma 1, i.e., it is the upper bound of the ratio of the mesh sizes of two consecutive grids.

REMARK 3.3. In the subsequent discussions, we will use the quantities $\eta_{s+1}^{(I)}(k_0)$, $\eta_{s+1}^{(II)}(k_0)$, $\eta_{s+1}(k_0)$, $\bar{\eta}(k_0)$, $\phi(k_0)$, and $\psi(k_0)$ defined in Lemma 7 as well as

$$\sigma(k_0) = [1+\phi(k_0)]\eta(k_0), \qquad \bar{\sigma}(k_0) = [1+\phi(k_0)]^2\eta(k_0),$$

$$\rho(k_0) = [1+\psi(k_0)]\eta(k_0), \qquad \bar{\rho}(k_0) = [1+\psi(k_0)]^2\eta(k_0),$$

$$\bar{\omega}(k_0) = \max\{\bar{\sigma}(k_0), \bar{\rho}(k_0)\}$$

in suitable places without further explanation. From Lemma 1 and Remark 3.2 it is evident that the following asymptotic behavior holds:

$$\bar{\omega}(k_0) = \begin{cases} O(k_0), & \text{for a 2D domain} \quad (q<1), \\ O(\mu^{k_0}), & \text{for a 3D domain} \quad (q<\mu^{-1}). \end{cases}$$

LEMMA 8. *Let $\tilde{M}^{(k)}$ for some fixed integer k $[sk_0 < k \leqslant (s+1)k_0,\ s = 0,1,\ldots,l(k_0)]$ be a SPD approximation to $B^{(k)}$ such that*

$$\lambda(B^{(k)-1}\tilde{M}^{(k)}) \in [1, 1+\tilde{\delta}_{k,s}]$$

holds for some $\tilde{\delta}_{k,s} \geqslant 0$. Define

$$M^{(k)} = \tilde{M}^{(k)},$$

and for $p = k+1, k+2, \ldots, k+k_0$ *set*

$$M^{(p)} = \begin{pmatrix} B_{11}^{(p)} & 0 \\ A_{21}^{(p)} & M^{(p-1)} \end{pmatrix} \begin{pmatrix} I & B_{11}^{(p)-1} A_{12}^{(p)} \\ 0 & I \end{pmatrix}.$$

Then

$$\lambda\left(B^{(k+k_0)-1} M^{(k+k_0)} \right) \in [1, 1+\delta_{k,s}]$$

with

$$\delta_{k,s} = \begin{cases} \tilde{\delta}_{k,s}\left[1 + \eta_{s+1}^{(\mathrm{I})}(k_0)\right]\eta(k_0) + \gamma^2 \sum_{j=1}^{k_0} \eta(j), & \textit{method (I)}, \\ \tilde{\delta}_{k,s}\left[1 + \eta_{s+1}^{(\mathrm{II})}(k_0)\right]\eta(k_0) + \gamma^2 \sum_{j=1}^{k_0} \eta(j) & \\ \quad + \eta_{s+1}^{(\mathrm{II})}(k_0)\left[1 + \eta_{s+1}^{(\mathrm{II})}(k_0)\right]\eta(k - sk_0), & \textit{method (II)}. \end{cases}$$

Here and in the subsequent discussion, we use $\lambda(\cdot)$ *to denote any eigenvalue of the corresponding matrix.*

Proof. First, for $i = k, k+1, \ldots, p$, let

$$v^{(i)} = \left(v_1^{(i)T}, v_2^{(i)T} \right)^T \in R^{n_i}, \qquad v_2^{(i)} = v^{(i-1)} \in R^{n_{i-1}}.$$

Noticing (3.6) and the definition of method (II), we easily see that

$$v^{(i)T}\left(B^{(i)} - \tilde{B}^{(i)} \right) v^{(i)} \geqslant 0, \qquad i = k, k+1, \ldots, p.$$

Since $B_{11}^{(i)}$ $(i = k, k+1, \ldots, p)$ are all SPD matrices, by Assumption (A_1), we see that

$$v_2^{(i)T} A_{21}^{(i)} B_{11}^{(i)-1} A_{12}^{(i)} v_2^{(i)} \geqslant 0, \qquad i = k, k+1, \ldots, p.$$

Additionally, by direct calculation, we can obtain

$$\begin{aligned} & v^{(p)T}\left(M^{(p)}-B^{(p)}\right)v^{(p)} \\ &\quad = v_2^{(p)T}A_{21}^{(p)}B_{11}^{(p)-1}A_{12}^{(p)}v_2^{(p)} \\ &\qquad + \begin{cases} v^{(p-1)T}\left(M^{(p-1)}-B^{(p-1)}\right)v^{(p-1)}, & \text{method (I)}, \\ v^{(p-1)T}\left(M^{(p-1)}-\tilde{B}^{(p-1)}\right)v^{(p-1)}, & \text{method (II)}. \end{cases} \end{aligned} \tag{3.12}$$

Therefore,

$$v^{(p)T}\left(M^{(p)}-B^{(p)}\right)v^{(p)} \geqslant v^{(p-1)T}\left(M^{(p-1)}-B^{(p-1)}\right)v^{(p-1)}$$

holds for both method (I) and method (II). By using this relation recursively and considering $v^{(k)T}(M^{(k)}-B^{(k)})v^{(k)} \geqslant 0$ under the condition, the inequality

$$v^{(p)T}\left(M^{(p)}-B^{(p)}\right)v^{(p)} \geqslant 0$$

can be obtained. In other words, there hold

$$\lambda\left(B^{(p)-1}M^{(p)}\right) \geqslant 1, \qquad p=k,k+1,\dots,k+k_0. \tag{3.13}$$

On the other hand, from (3.12) we have

$$\begin{aligned} & v^{(p)T}\left(M^{(p)}-B^{(p)}\right)v^{(p)} \\ &\quad = v^{(p-1)T}\left(M^{(p-1)}-B^{(p-1)}\right)v^{(p-1)} + v_2^{(p)T}A_{21}^{(p)}B_{11}^{(p)-1}A_{12}^{(p)}v_2^{(p)} \\ &\qquad + \begin{cases} 0, & \text{method (I)}, \\ v^{(p-1)T}\left(B^{(p-1)}-\tilde{B}^{(p-1)}\right)v^{(p-1)}, & \text{method (II)}, \end{cases} \end{aligned} \tag{3.14}$$

for $p=k,k+1,\dots,k+k_0$. Furthermore, according to Assumption (A_1) and Lemma 4 there hold

$$\begin{aligned} v_2^{(p)T}A_{21}^{(p)}B_{11}^{(p)-1}A_{12}^{(p)}v_2^{(p)} &\leqslant v_2^{(p)T}A_{21}^{(p)}A_{11}^{(p)-1}A_{12}^{(p)}v_2^{(p)} \\ &\leqslant \gamma^2 v^{(p-1)T}A^{(p-1)}v^{(p-1)}. \end{aligned}$$

By the definition of method (II), we have

$$v^{(p-1)T}\left(B^{(p-1)}-\tilde{B}^{(p-1)}\right)v^{(p-1)}=0$$

for $p \neq (s+1)k_0 + 1$, and in light of Lemma 7 and (3.6) we then get

$$\begin{aligned}
& v^{((s+1)k_0)T}\left(B^{((s+1)k_0)} - \tilde{B}^{((s+1)k_0)}\right)v^{((s+1)k_0)} \\
&\quad = v^{((s+1)k_0)T}\left(B^{((s+1)k_0)} - A^{((s+1)k_0)}\right)v^{((s+1)k_0)} \\
&\quad \leqslant \eta_{s+1}(k_0)\, v^{((s+1)k_0)T}A^{((s+1)k_0)}v^{((s+1)k_0)} \\
&\quad \leqslant \eta_{s+1}(k_0)\, v^{((s+1)k_0)T}B^{((s+1)k_0)}v^{((s+1)k_0)}.
\end{aligned}$$

Therefore, the estimates

$$\begin{aligned}
& v^{(p)T}\left(M^{(p)} - B^{(p)}\right)v^{(p)} \\
&\quad \leqslant v^{(k)T}\left(M^{(k)} - B^{(k)}\right)v^{(k)} + \gamma^2 \sum_{j=k}^{p-1} v^{(j)T}A^{(j)}v^{(j)} \\
&\qquad + \begin{cases} 0, & \text{method (I)}, \\ \eta_{s+1}(k_0)\, v^{((s+1)k_0)T}B^{((s+1)k_0)}v^{((s+1)k_0)}, & \text{method (II)}, \end{cases}
\end{aligned} \tag{3.15}$$

can be obtained through regressively using (3.14). Additionally, using Lemma 1 and (3.6), we can get

$$\begin{aligned}
\sum_{j=k}^{k+k_0-1} v^{(j)T}A^{(j)}v^{(j)} &\leqslant \sum_{j=k}^{k+k_0-1} \eta(k + k_0 - j)\, v^{(k+k_0)T}A^{(k+k_0)}v^{(k+k_0)} \\
&\leqslant \sum_{j=1}^{k_0} \eta(j)\, v^{(k+k_0)T}B^{(k+k_0)}v^{(k+k_0)}
\end{aligned} \tag{3.16}$$

as well as

$$\begin{aligned}
& v^{((s+1)k_0)T}B^{((s+1)k_0)}v^{((s+1)k_0)} \\
&\quad \leqslant \left[1 + \eta_{s+1}(k_0)\right] v^{((s+1)k_0)T}A^{((s+1)k_0)}v^{((s+1)k_0)} \\
&\quad \leqslant \left[1 + \eta_{s+1}(k_0)\right] \eta(k - sk_0)\, v^{(k+k_0)T}A^{(k+k_0)}v^{(k+k_0)} \\
&\quad \leqslant \left[1 + \eta_{s+1}(k_0)\right] \eta(k - sk_0)\, v^{(k+k_0)T}B^{(k+k_0)}v^{(k+k_0)},
\end{aligned} \tag{3.17}$$

where we have used Lemma 7, Lemma 1, and (3.6) in each of the three inequalities successively. Now, substituting (3.16)–(3.17) into (3.15) with $p = k + k_0$, we immediately obtain

$$v^{(k+k_0)T}\left(M^{(k+k_0)} - B^{(k+k_0)}\right)v^{(k+k_0)} \leqslant \delta_{k,s} v^{(k+k_0)T} B^{(k+k_0)} v^{(k+k_0)}.$$

This is just the conclusion of Lemma 8. Here, the estimation

$$\begin{aligned} v^{(k)T}B^{(k)}v^{(k)} &\leqslant \left[1 + \eta_{s+1}(k_0)\right] v^{(k)T}A^{(k)}v^{(k)} \\ &\leqslant \left[1 + \eta_{s+1}(k_0)\right]\eta(k_0)\, v^{(k+k_0)T}A^{(k+k_0)}v^{(k+k_0)} \\ &\leqslant \left[1 + \eta_{s+1}(k_0)\right]\eta(k_0)\, v^{(k+k_0)T}B^{(k+k_0)}v^{(k+k_0)}, \end{aligned}$$

resulting from Lemma 7, Lemma 1, and (3.6), has been considered. ■

4. MAIN RESULTS

We first give a general estimation of the relative condition numbers of $M^{(sk_0)}$ with respect to $A^{(sk_0)}$ for $s = 1, 2, \ldots, l(k_0)$.

THEOREM 4.1. *Let Assumptions* (A_1)–(A_2) *and the condition* (3.3) *be satisfied, and define*

$$\lambda^{(s)} = \sup_{v \neq 0} \frac{v^T M^{(sk_0)} v}{v^T A^{(sk_0)} v} \qquad \left(\lambda^{(0)} \equiv 1\right),$$

$$\alpha^{(s)} = \begin{cases} (1 - \gamma^2)/\lambda^{(s)}, & \textit{version } (\mathrm{i}), \\ 1/\lambda^{(s)}, & \textit{version } (\mathrm{ii}), \end{cases}$$

$$s = 1, 2, \ldots, l(k_0). \quad (4.1)$$

Then there hold

$$\lambda^{(s+1)} \leqslant \left[1 + \eta_{s+1}(k_0)\right]\left(1 + \delta_{sk_0,s}\right), \qquad s = 0, 1, \ldots, l/k_0 - 2 \quad (4.2)$$

with

$$\delta_{sk_0,s} = \begin{cases} \dfrac{p_\nu(\alpha^{(s)})}{1-p_\nu(\alpha^{(s)})}\left[1+\eta_{s+1}^{(\mathrm{I})}(k_0)\right]\eta(k_0) + \gamma^2 \sum\limits_{j=1}^{k_0} \eta(j), & \textit{method } (\mathrm{I}), \\ \dfrac{p_\nu(\alpha^{(s)})}{1-p_\nu(\alpha^{(s)})}\left[1+\eta_{s+1}^{(\mathrm{II})}(k_0)\right]\eta(k_0) + \gamma^2 \sum\limits_{j=1}^{k_0} \eta(j) & \\ \quad + \eta_{s+1}^{(\mathrm{II})}(k_0)\left[1+\eta_{s+1}^{(\mathrm{II})}(k_0)\right], & \textit{method } (\mathrm{II}). \end{cases} \tag{4.3}$$

Proof. For all $s \in \{1, 2, \ldots, l(k_0)\}$, let

$$\bar{\lambda}^{(s)} = \sup_{v \neq 0} \frac{v^T M^{(sk_0)} v}{v^T B^{(sk_0)} v} \quad (\bar{\lambda}^{(0)} \equiv 1), \qquad \tilde{\lambda}^{(s)} = \sup_{v \neq 0} \cdot \frac{v^T \tilde{M}^{(sk_0)} v}{v^T B^{(sk_0)} v} \quad (\tilde{\lambda}^{(0)} \equiv 1).$$

From (2.10) we immediately have

$$u^T \tilde{S}^{(sk_0+1)} u \leqslant u^T B^{(sk_0)} u \quad \forall u \in R^{n_{sk_0}}, \qquad s = 1, 2, \ldots, l(k_0).$$

Additionally, by making use of (3.6). Assumption (A_1), and Lemma 4, we can get

$$(1-\gamma^2) u^T A^{(sk_0)} u \leqslant u^T S^{(sk_0+1)} u \leqslant u^T \tilde{S}^{(sk_0+1)} u \qquad \forall u \in R^{n_{sk_0}},$$
$$s = 1, 2, \ldots, l(k_0).$$

Now, if we denote

$$S_s = \begin{cases} \tilde{S}^{(sk_0+1)}, & \text{version (i)}, \\ B^{(sk_0)}, & \text{version (ii)}, \end{cases} \tag{4.4}$$

then clearly

$$u^T B^{(sk_0)} u \geqslant u^T S_s u \geqslant \begin{cases} (1-\gamma^2) u^T A^{(sk_0)} u, & \text{version (i)}, \\ u^T B^{(sk_0)} u, & \text{version (ii)}, \end{cases} \tag{4.5}$$

for $\forall u \in R^{n_{sk_0}}$, $\forall s \in \{1, 2, \dots, l(k_0)\}$, and

$$\hat{M}^{(sk_0)} = S_s\left[I - p_\nu\left(M^{(sk_0)-1} S_s\right)\right]^{-1}, \qquad s = 1, 2, \dots, l(k_0). \tag{4.6}$$

Write

$$T_s = S_s^{1/2} M^{(sk_0)-1} S_s^{1/2}, \qquad s = 1, 2, \dots, l(k_0), \tag{4.7}$$

we can assert that

$$\lambda\left(B^{(sk_0)-1} M^{(sk_0)}\right) \in [1, +\infty) \tag{4.8}$$

$$\lambda(T_s) \in [\alpha^{(s)}, 1] \tag{4.9}$$

for $s = 1, 2, \dots, l(k_0)$.

As a matter of fact, remembering (3.6) and by direct calculations, the relation

$$\begin{aligned} u^{(p)T}\left(M^{(p)} - B^{(p)}\right) u^{(p)} &\geqslant u^{(p-1)T}\left(M^{(p-1)} - B^{(p-1)}\right) u^{(p-1)} \\ &\quad + u^{(p-1)T} A_{21}^{(p)} B_{11}^{(p)-1} A_{12}^{(p)} u^{(p-1)} \end{aligned} \tag{4.10}$$

can be obtained from (2.7) as well as the definitions of method (I) and method (II) for all $u^{(p)} = (u_1^{(p)T}, u^{(p-1)T})^T \in R^{n_p}$. Based on this relation, (4.8)–(4.9) can be derived by induction:

When $s = 1$, considering (2.8) and Assumption (A_1), we can easily get

$$u^{(k_0)T}\left(M^{(k_0)} - B^{(k_0)}\right) u^{(k_0)} \geqslant u^{(k_0-1)T}\left(M^{(k_0-1)} - B^{(k_0-1)}\right) u^{(k_0-1)} \geqslant \cdots \geqslant 0,$$

i.e., (4.8) holds in this case. Since by (4.5) we have

$$\sup_{u\neq 0}\frac{u^T T_1 u}{u^T u} \leqslant \sup_{u\neq 0}\frac{u^T S_1 u}{u^T M^{(k_0)} u} \leqslant \sup_{u\neq 0}\frac{u^T B^{(k_0)} u}{u^T M^{(k_0)} u} \leqslant 1, \tag{4.11}$$

$$\inf_{u\neq 0}\frac{u^T T_1 u}{u^T u} \geqslant \inf_{u\neq 0}\frac{u^T S_1 u}{u^T M^{(k_0)} u} \geqslant \begin{cases} (1-\gamma^2)\inf_{u\neq 0}\dfrac{u^T A^{(k_0)} u}{u^T M^{(k_0)} u}, & \text{version (i)}, \\ \inf_{u\neq 0}\dfrac{u^T A^{(k_0)} u}{u^T M^{(k_0)} u}, & \text{version (ii)} \end{cases}$$

$$= \begin{cases} (1-\gamma^2)/\lambda^{(1)}, & \text{version (i)}, \\ 1/\lambda^{(1)}, & \text{version (ii)} \end{cases}$$

$$= \alpha^{(1)}, \tag{4.12}$$

(4.9) holds for $s = 1$ too. Furthermore, from (2.8)–(2.9) and by using (4.6) we see that

$$\begin{aligned} & u^{(k_0+1)T}\left(M^{(k_0+1)} - B^{(k_0+1)}\right)u^{(k_0+1)} \\ & \geqslant u^{(k_0)T}\left(\hat{M}^{(k_0)} - S_1\right)u^{(k_0)} \\ & = u^{(k_0)T} S_1^{1/2}\left\{\left[I - p_\nu(T_1)\right]^{-1} - I\right\} S_1^{1/2} u^{(k_0)} \geqslant 0. \end{aligned}$$

Now suppose that (4.8)–(4.9) hold for some $s \in \{1, 2, \dots, l(k_0)\}$. We can similarly obtain

$$u^{(sk_0+1)T}\left(M^{(sk_0+1)} - B^{(sk_0+1)}\right)u^{(sk_0+1)} \geqslant 0$$

and hence

$$u^{((s+1)k_0)T}\left(M^{((s+1)k_0)} - B^{((s+1)k_0)}\right)u^{((s+1)k_0)} \geqslant 0$$

by recursively using (4.10). Applying (4.5) again, through derivations analogous to (4.11)–(4.12) we can also get $\lambda(T_{s+1}) \in [\alpha^{(s+1)}, 1]$.

By induction, (4.8)–(4.9) have been proved completely.

On the other hand, by direct computation we have

$$\begin{aligned}
\tilde{\lambda}^{(s)} &= \sup_{u \neq 0} \frac{u^T S_s \left[I - p_\nu\left(M^{(sk_0)^{-1}} S_s\right)\right]^{-1} u}{u^T B^{(sk_0)} u} \\
&\leqslant \sup_{v \neq 0} \frac{v^T \left[I - p_\nu(T_s)\right]^{-1} v}{v^T v} \sup_{v \neq 0} \frac{v^T S_s v}{v^T B^{(sk_0)} v} \\
&\leqslant \sup_{t \in [\alpha^{(s)}, 1]} \frac{1}{1 - p_\nu(t)} \\
&= \frac{1}{1 - p_\nu(\alpha^{(s)})};
\end{aligned}$$

here we have used the inequality (4.5) and Assumption (A_2). In accordance with Lemma 8, there obviously holds

$$\bar{\lambda}^{(s+1)} \leqslant 1 + \delta_{sk_0, s}.$$

Considering Lemma 7, we finally get

$$\lambda^{(s+1)} \leqslant \left[1 + \eta_{s+1}(k_0)\right] \bar{\lambda}^{(s+1)} \leqslant \left[1 + \eta_{s+1}(k_0)\right]\left(1 + \delta_{sk_0, s}\right). \qquad \blacksquare$$

For convenience in the subsequent expressions, we introduce the following notation:

$$\xi(k_0) = 1 + \gamma^2 \sum_{j=1}^{k_0} \eta(j) - \left[1 + \psi(k_0)\right] \eta(k_0),$$

$$\zeta(k_0) = 1 + \gamma^2 \sum_{j=1}^{k_0} \eta(j) + \left[1 + \phi(k_0)\right]\left[\phi(k_0) - \eta(k_0)\right],$$

$$\bar{\xi}(k_0) = \left[1 + \psi(k_0)\right] \xi(k_0),$$

$$\bar{\zeta}(k_0) = \left[1 + \phi(k_0)\right] \zeta(k_0),$$

which will also be used throughout the remainder of this paper.

THEOREM 4.2. *Let Assumptions* (A_1)–(A_2) *and the condition* (3.3) *be satisfied, and assume that* $q^{k_0}\eta(k_0) < 1$ *holds for method* (I). *If we define*

$$\hat{\lambda}^{(1)} = [1 + \bar{\eta}(k_0)]\left(1 + \hat{\delta}_{0,0}\right), \qquad \hat{\lambda}^{(s+1)} = [1 + \bar{\eta}(k_0)]\left(1 + \hat{\delta}_{sk_0,s}\right),$$

$$\hat{\alpha}^{(s)} = \begin{cases} (1-\gamma^2)/\hat{\lambda}^{(s)}, & \textit{version (i)}, \\ 1/\hat{\lambda}^{(s)}, & \textit{version (ii)}, \end{cases} \qquad s = 1, 2, \ldots, l(k_0), \tag{4.13}$$

with

$$\hat{\alpha}^{(0)} = \alpha^{(0)}$$

$$\hat{\delta}_{sk_0,s} = \begin{cases} \dfrac{p_\nu(\hat{\alpha}^{(s)})}{1 - p_\nu(\hat{\alpha}^{(s)})}\rho(k_0) + \gamma^2 \sum\limits_{j=1}^{k_0} \eta(j), & \textit{method (I)}, \\ \dfrac{p_\nu(\hat{\alpha}^{(s)})}{1 - p_\nu(\hat{\alpha}^{(s)})}\sigma(k_0) + \gamma^2 \sum\limits_{j=1}^{k_0} \eta(j) + \phi(k_0)[1 + \phi(k_0)], \\ & \textit{method (II)}, \end{cases}$$

$$s = 1, 2, \ldots, l(k_0), \tag{4.14}$$

then $\{\hat{\lambda}^{(s)}\}$ *becomes a majorizing sequence of* $\{\lambda^{(s)}\}$. *Now*:

(a) *For version* (i) *there hold*

$$\hat{\alpha}^{(s+1)} = \begin{cases} \dfrac{(1-\gamma^2)\hat{\alpha}^{(s)} Q_{\nu-1}(\hat{\alpha}^{(s)})}{\bar{\rho}(k_0) + \bar{\xi}(k_0)\hat{\alpha}^{(s)} Q_{\nu-1}(\hat{\alpha}^{(s)})}, & \textit{method (I)}, \\ \dfrac{(1-\gamma^2)\hat{\alpha}^{(s)} Q_{\nu-1}(\hat{\alpha}^{(s)})}{\bar{\sigma}(k_0) + \bar{\zeta}(k_0)\hat{\alpha}^{(s)} Q_{\nu-1}(\hat{\alpha}^{(s)})}, & \textit{method (II)}, \end{cases}$$

$$s = 0, 1, 2, \ldots, l/k_0 - 3. \tag{4.15}$$

Therefore, when

$$\begin{aligned} &\bar{\rho}(k_0) < (1-\gamma^2)Q_{\nu-1}(0), \quad q^{k_0}\eta(k_0) < 1, && \textit{method (I)}, \\ &\bar{\sigma}(k_0) < (1-\gamma^2)Q_{\nu-1}(0), && \textit{method (II)}, \end{aligned} \tag{4.16}$$

each of the sequences $\{\hat{\alpha}^{(s)}\}_{s=0}^{\infty}$ *defined by* (4.15) *has a unique limit point* $\alpha^* \in (0,1)$ *such that*

$$\hat{\alpha}^{(s)} \geqslant \alpha^*, \qquad s = 0, 1, 2, \ldots, \tag{4.17}$$

where $\alpha^* \in (0,1)$ *is the smallest positive root corresponding to the following equations*:

$$\begin{aligned} &(1-\gamma^2)Q_{\nu-1}(t) - \bar{\xi}(k_0)tQ_{\nu-1}(t) - \bar{\rho}(k_0) = 0, && \textit{method (I)},\\ &(1-\gamma^2)Q_{\nu-1}(t) - \bar{\zeta}(k_0)tQ_{\nu-1}(t) - \bar{\sigma}(k_0) = 0, && \textit{method (II)}. \end{aligned} \tag{4.18}$$

(b) *For version* (ii) *there hold*

$$\hat{\alpha}^{(s+1)} = \begin{cases} \dfrac{\hat{\alpha}^{(s)}Q_{\nu-1}(\hat{\alpha}^{(s)})}{\bar{\rho}(k_0) + \bar{\xi}(k_0)\hat{\alpha}^{(s)}Q_{\nu-1}(\hat{\alpha}^{(s)})}, & \textit{method (I)},\\[2ex] \dfrac{\hat{\alpha}^{(s)}Q_{\nu-1}(\hat{\alpha}^{(s)})}{\bar{\sigma}(k_0) + \bar{\zeta}(k_0)\hat{\alpha}^{(s)}Q_{\nu-1}(\hat{\alpha}^{(s)})}, & \textit{method (II)}, \end{cases}$$

$$s = 0, 1, 2, \ldots, l/k_0 - 3. \tag{4.19}$$

Therefore, when

$$\begin{aligned} &\bar{\rho}(k_0) < Q_{\nu-1}(0), \quad q^{k_0}\eta(k_0) < 1, && \textit{method (I)},\\ &\bar{\sigma}(k_0) < Q_{\nu-1}(0), && \textit{method (II)}, \end{aligned} \tag{4.20}$$

each of the sequences $\{\hat{\alpha}^{(s)}\}_{s=0}^{\infty}$ *defined by* (4.19) *has a unique limit point* $\alpha^* \in (0,1)$ *such that*

$$\hat{\alpha}^{(s)} \geqslant \alpha^*, \qquad s = 0, 1, 2, \ldots, \tag{4.21}$$

where $\alpha^* \in (0,1)$ *is the smallest positive root corresponding to the following equations*:

$$\begin{aligned} Q_{\nu-1}(t) - \bar{\xi}(k_0)tQ_{\nu-1}(t) - \bar{\rho}(k_0) &= 0, &\quad \textit{method (I)},\\ Q_{\nu-1}(t) - \bar{\zeta}(k_0)tQ_{\nu-1}(t) - \bar{\sigma}(k_0) &= 0, &\quad \textit{method (II)}. \end{aligned} \tag{4.22}$$

Proof. We first demonstrate by induction that $\{\hat{\lambda}^{(s)}\}$ is a majorizing sequence of $\{\lambda^{(s)}\}$, i.e.,

$$\lambda^{(s)} \leqslant \hat{\lambda}^{(s)}, \qquad s = 1, 2, \dots. \tag{4.23}$$

Noticing that $\delta_{0,0} \leqslant \hat{\delta}_{0,0}$, for $s = 1$ we clearly have $\lambda^{(1)} \leqslant \hat{\lambda}^{(1)}$ by the definition. Now, we assume that $\lambda^{(s)} \leqslant \hat{\lambda}^{(s)}$ has been got for some s; then it is easy to see that $\alpha^{(s)} \geqslant \hat{\alpha}^{(s)}$. Recalling the monotone nonincreasing property of the polynomial $p_\nu(t)$ in $[0,1]$, we immediately know $p_\nu(\hat{\alpha}^{(s)}) \geqslant p_\nu(\alpha^{(s)})$. Therefore, the inequality $\delta_{sk_0,s} \leqslant \hat{\delta}_{sk_0,s}$ can be derived. Hence, $\lambda^{(s+1)} \leqslant \hat{\lambda}^{(s+1)}$. Making use of the inductive principle, (4.23) is completely confirmed now.

Based on (4.13) and (4.14), we can easily obtain that $\{\hat{\alpha}^{(s)}\}$ satisfies the recurrence relations (4.15) and (4.19), respectively.

In the following, we will use the case of version (i) of method (I) as an example to show the remainder of the proof. The demonstrations of the other cases are very similar to that of this one.

We use induction to determine the positive number α^* which makes (4.17) hold uniformly under the condition (4.16). In fact, if we assume for some s that the inequality $\hat{\alpha}^{(s)} \geqslant \alpha^*$ has been obtained, then, in order to get $\hat{\alpha}^{(s+1)} \geqslant \alpha^*$, by the first relation of (4.15) we only need to demonstrate that

$$1 - \gamma^2 \geqslant \frac{\bar{\rho}(k_0) + \bar{\xi}(k_0)\alpha^* Q_{\nu-1}(\alpha^*)}{Q_{\nu-1}(\alpha^*)} \tag{4.24}$$

is valid. Define a one-variable function

$$f(t) = \frac{\bar{\rho}(k_0)}{Q_{\nu-1}(t)} + \bar{\xi}(k_0)t. \tag{4.25}$$

By noticing $f(1) > 1$, we know that (4.24) holds only if

$$1 - \gamma^2 > \lim_{t \to 0} f(t). \tag{4.26}$$

Since

$$\lim_{t \to 0} f(t) = \frac{\bar{\rho}(k_0)}{Q_{\nu-1}(0)},$$

by substituting this identity into (4.26) it can be immediately seen that there exists $\alpha^* \in (0,1)$ such that (4.17) holds provided the first inequality in (4.16) is satisfied. From (4.24), α^* can be taken to be the smallest positive real number which makes (4.24) become to equality. This shows that $\alpha^* \in (0,1)$ is the unique limit point of the sequence $\{\hat{\alpha}^{(s)}\}_{s=0}^{\infty}$ and satisfies the first equation of (4.18). ∎

Theorem 4.2 obviously implies the following conclusion.

THEOREM 4.3. *Let Assumptions* (A_1) *and* (A_2) *be satisfied. Then:*

(a) *For version* (i) *there holds*

$$\lambda(A^{(sk_0)^{-1}} M^{(sk_0)}) \in \left[1, \frac{1-\gamma^2}{\alpha^*}\right], \qquad s = 1, 2, \dots, l(k_0),$$

provided (4.16) *is satisfied correspondingly. Here* $\alpha^* \in (0,1)$ *is the smallest positive root of Equation* (4.18).

(b) *For version* (ii) *there holds*

$$\lambda(A^{(sk_0)^{-1}} M^{(sk_0)}) \in \left[1, \frac{1}{\alpha^*}\right], \qquad s = 1, 2, \dots, l(k_0),$$

provided (4.20) *is satisfied correspondingly. Here* $\alpha^* \in (0,1)$ *is the smallest positive root of Equation* (4.22).

We now specialize the above theoretical result to the two polynomials defined by (2.5)–(2.6) to get applicable bounds for the relative condition numbers of $M^{(sk_0)}$ with respect to $A^{(sk_0)}$ [$s = 1, 2, \dots, l(k_0)$].

THEOREM 4.4. *Let Assumption* (A_1) *be satisfied, and the polynomial* $p_\nu(t)$ $(0 \leqslant t \leqslant 1)$ *be given by* (2.5). *Then*:

(a) *For version* (i), *if*

$$\frac{\bar{\rho}(k_0)}{2\nu} \frac{\left[(1+\sqrt{\alpha})^\nu + (1-\sqrt{\alpha})^\nu\right]^2}{\sum_{j=0}^{(\nu-1)/2} \binom{\nu}{2j+1} \alpha^j (1-\alpha)^{\nu-1}} < 1-\gamma^2, \quad q^{k_0}\eta(k_0) < 1, \qquad \textit{method } (\mathrm{I}),$$

$$\frac{\bar{\sigma}(k_0)}{2\nu} \frac{\left[(1+\sqrt{\alpha})^\nu + (1-\sqrt{\alpha})^\nu\right]^2}{\sum_{j=0}^{(\nu-1)/2} \binom{\nu}{2j+1} \alpha^j (1-\alpha)^{\nu-1}} < 1-\gamma^2, \quad \textit{method } (\mathrm{II}), \tag{4.27}$$

there hold

$$\lambda\left(A^{(sk_0)-1}M^{(sk_0)}\right) \in \left[1, \frac{1-\gamma^2}{\alpha}\right], \qquad s = 1, 2, \ldots, l(k_0), \tag{4.28}$$

where $\alpha \in (0,1)$ *is the smallest positive root of the following equation*:

$$\left(\frac{(1+\sqrt{t})^\nu + (1-\sqrt{t})^\nu}{2\sum_{j=0}^{(\nu-1)/2} \binom{\nu}{2j+1} t^j}\right)^2 = \begin{cases} \dfrac{1-\gamma^2-\bar{\xi}(k_0)t}{\bar{\rho}(k_0)}, & \textit{method } (\mathrm{I}), \\[2ex] \dfrac{1-\gamma^2-\bar{\zeta}(k_0)t}{\bar{\sigma}(k_0)}, & \textit{method } (\mathrm{II}). \end{cases} \tag{4.29}$$

Moreover, once

$$\nu^2 > \frac{2\bar{\rho}(k_0)}{1-\gamma^2}, \quad q^{k_0}\eta(k_0) < 1, \qquad \textit{method } (\mathrm{I}),$$

$$\nu^2 > \frac{2\bar{\sigma}(k_0)}{1-\gamma^2}, \qquad \textit{method } (\mathrm{II}),$$

the smallest solution $\alpha \in (0,1)$ *of the equation* (4.29) *can guarantee that* (4.27) *holds, and therefore that* (4.28) *holds.*

(b) *For version* (ii), *if*

$$\begin{aligned} &\frac{\bar{\rho}(k_0)}{2\nu}\frac{\left[(1+\sqrt{\alpha})^{\nu}+(1-\sqrt{\alpha})^{\nu}\right]^2}{\sum_{j=0}^{(\nu-1)/2}\binom{\nu}{2j+1}\alpha^j(1-\alpha)^{\nu-1}} < 1, \quad q^{k_0}\eta(k_0) < 1, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{method } (\mathrm{I}), \\ &\frac{\bar{\sigma}(k_0)}{2\nu}\frac{\left[(1+\sqrt{\alpha})^{\nu}+(1-\sqrt{\alpha})^{\nu}\right]^2}{\sum_{j=0}^{(\nu-1)/2}\binom{\nu}{2j+1}\alpha^j(1-\alpha)^{\nu-1}} < 1, \quad \textit{method } (\mathrm{II}), \end{aligned} \tag{4.30}$$

there hold

$$\lambda\left(A^{(sk_0)^{-1}}M^{(sk_0)}\right) \in [1, 1/\alpha], \qquad s = 1, 2, \ldots, l(k_0), \tag{4.31}$$

where $\alpha \in (0,1)$ *is the smallest positive root of the following equation*:

$$\left(\frac{(1+\sqrt{t})^{\nu}+(1-\sqrt{t})^{\nu}}{2\sum_{j=0}^{(\nu-1)/2}\binom{\nu}{2j+1}t^j}\right)^2 = \begin{cases} \dfrac{1-\bar{\xi}(k_0)t}{\bar{\rho}(k_0)}, & \textit{method } (\mathrm{I}), \\ \dfrac{1-\bar{\zeta}(k_0)t}{\bar{\sigma}(k_0)}, & \textit{method } (\mathrm{II}). \end{cases} \tag{4.32}$$

Moreover, once

$$\begin{aligned} &\nu^2 > 2\bar{\rho}(k_0), \quad q^{k_0}\eta(k_0) < 1, \qquad && \textit{method } (\mathrm{I}), \\ &\nu^2 > 2\bar{\sigma}(k_0), && \textit{method } (\mathrm{II}), \end{aligned}$$

the smallest solution $\alpha \in (0,1)$ *of the equation* (4.32) *can guarantee that* (4.30) *holds, and therefore that* (4.31) *holds.*

THEOREM 4.5. *Let Assumption* (A_1) *be satisfied, and the polynomial* $p_\nu(t)$ $(0 \leqslant t \leqslant 1)$ *be given by* (2.6). *Then*:

(a) *For version* (i), *if*

$$\begin{aligned} &\nu > \frac{\bar{\rho}(k_0)}{1-\gamma^2}, \quad q^{k_0}\eta(k_0) < 1, && \textit{method (I)}, \\ &\nu > \frac{\bar{\sigma}(k_0)}{1-\gamma^2}, && \textit{method (II)}, \end{aligned}$$

there hold

$$\lambda\left(A^{(sk_0)-1}M^{(sk_0)}\right) \in \left[1, \frac{1-\gamma^2}{\alpha}\right], \qquad s = 1,2,\ldots,l(k_0),$$

where $\alpha \in (0,1)$ *is the smallest positive root of the following equation*:

$$\sum_{j=1}^{\nu}(-1)^j\binom{\nu}{j}t^{j-1} = \begin{cases} \dfrac{\bar{\rho}(k_0)}{1-\gamma^2-\bar{\xi}(k_0)t}, & \textit{method (I)}, \\[2ex] \dfrac{\bar{\sigma}(k_0)}{1-\gamma^2-\bar{\zeta}(k_0)t}, & \textit{method (II)}. \end{cases}$$

(b) *For version* (ii), *if*

$$\begin{aligned} &\nu > \bar{\rho}(k_0), \quad q^{k_0}\eta(k_0) < 1, && \textit{method (I)}, \\ &\nu > \bar{\sigma}(k_0), && \textit{method (II)}, \end{aligned}$$

there hold

$$\lambda\left(A^{(sk_0)-1}M^{(sk_0)}\right) \in [1, 1/\alpha], \qquad s = 1,2,\ldots,l(k_0),$$

where $\alpha \in (0,1)$ is the smallest positive root of the following equation:

$$\sum_{j=1}^{\nu} (-1)^j \binom{\nu}{j} t^{j-1} = \begin{cases} \dfrac{\bar{\rho}(k_0)}{1-\bar{\xi}(k_0)t}, & \textit{method (I)}, \\ \dfrac{\bar{\sigma}(k_0)}{1-\bar{\zeta}(k_0)t}, & \textit{method (II)}. \end{cases}$$

5. COMPUTATIONAL COMPLEXITY

We now consider the asymptotic work estimation in each iterative step of our new hybrid algebraic multilevel preconditioning methods. Without loss of generality, we assume the refinements are uniform. Then the number of nodes n_k at the kth discretization level grows in geometrical fashion, i.e.,

$$n_k = n_l \mu^{d(k-1)}, \qquad k = 1, 2, \dots, l, \tag{5.1}$$

with $d = 2$ or $d = 3$, respectively. We recall that $\mu \geqslant \max_{1 \leqslant k \leqslant l-1} h_k / h_{k+1} \geqslant 2$ (h_k is the discretization parameter at level k).

Let $W(s)$ be the amount of arithmetic work performed on level sk_0. Then we have for $(s-1)k_0 + 1 \leqslant k \leqslant sk_0$ that

$$W_k \leqslant c_1(\mu^d - 1) n_{k-1} + W_{k-1},$$

where W_k denotes the amount of arithmetic work performed on level k. Hence

$$W_k \leqslant c_1(1 - \mu^{-d}) n_k \left[1 + \mu^{-d} + \mu^{-2d} + \cdots + \mu^{-(k-j-1)d}\right] W_j$$

holds for j $[(s-1)k_0 + 1 \leqslant j \leqslant sk_0]$. From this inequality we can directly obtain

$$W(s) \leqslant \bar{c} n_{sk_0} + W_{(s-1)k_0+1}, \qquad \bar{c} = c_1\left(1 - \mu^{-(k_0-1)d}\right).$$

Considering

$$\begin{aligned} W_{(s-1)k_0+1} &\leqslant c_1(\nu - 1)(\mu^d - 1) n_{(s-1)k_0} + \nu W(s-1) \\ &\leqslant \bar{c} n_{sk_0} + \nu W(s-1) \end{aligned}$$

with

$$\tilde{c} = c_1(\nu - 1)(\mu^d - 1)\mu^{-k_0 d},$$

we know that there holds

$$W(s+1) \leqslant \nu W(s) + c n_{(s+1)k_0}, \tag{5.2}$$

where

$$c = \bar{c} + \tilde{c} = c_1\{1 + \mu^{-k_0 d}[(\nu - 2)(\mu^d - 2) - 1]\}. \tag{5.3}$$

By using (5.2) recursively, we have

$$\begin{aligned} W(s+1) &\leqslant c \sum_{j=0}^{s-1} \nu^j n_{(s-j+1)k_0} + \nu^s W(1) \\ &= c \sum_{j=0}^{s-1} \nu^j \mu^{d(s-j+1)k_0 - l} n_l + \nu^s W(1) \\ &= c n_l \mu^{d(s+1)k_0 - l} \sum_{j=0}^{s-1} \left(\nu \mu^{-k_0 d}\right)^j + \nu^s W(1) \\ &\leqslant n_{(s+1)k_0}\left[c \sum_{j=0}^{s-1} \left(\nu\mu^{-k_0 d}\right)^j + \frac{W(1)}{n_{k_0}}\left(\nu\mu^{-k_0 d}\right)^s\right]. \end{aligned}$$

Then, if $\nu\mu^{-k_0 d} < 1$, we get

$$\frac{W(s+1)}{n_{(s+1)k_0}} \leqslant c^* + \frac{W(1)}{n_{k_0}}.$$

That is, the asymptotic work estimation shows that the new hybrid algebraic multilevel preconditioners wil! be of optimal order provided ν satisfies the inequalities

$$Q_{\nu-1}(0) > \begin{cases} \bar{\omega}(k_0)/(1-\gamma^2), & \text{version (i)}, \\ \bar{\omega}(k_0), & \text{version (ii)}, \end{cases}$$

from Theorem 4.2, and

$$\nu\mu^{-k_0 d} < 1$$

from the complexity requirement. More concretely, for the polynomials defined by (2.5), based on the asymptotic behavior of $\overline{\omega}(k_0)$ (see Remark 3.3), we know that these restrictions on ν become

$$\mu^{k_0 d} > \nu > \begin{cases} \sqrt{\dfrac{2}{1-\gamma^2}}\sqrt{\overline{\omega}(k_0)}\,, & \text{version (i)}, \\ \sqrt{2\overline{\omega}(k_0)}\,, & \text{version (ii)}, \end{cases}$$

$$= \begin{cases} O\left(\sqrt{k_0}\right), & d = 2 \text{ for a plane polygon}, \\ O(\mu^{k_0/2}), & d = 3 \text{ for a 3D polytope}, \end{cases}$$

while for the polynomial defined by (2.6), also based on the asymptotic behavior of $\overline{\omega}(k_0)$, we see that the above restrictions on ν become

$$\mu^{k_0 d} > \nu > \begin{cases} \overline{\omega}(k_0)/(1-\gamma^2), & \text{version (i)}, \\ \overline{\omega}(k_0), & \text{version (ii)} \end{cases}$$

$$= \begin{cases} O(k_0), & d = 2 \text{ for a plane polygon}, \\ O(\mu^{k_0}), & d = 3 \text{ for a 3D polytope}. \end{cases}$$

It is clear now that asymptotically, for k_0 sufficiently large, the restrictions on ν corresponding to the polynomials (2.5)–(2.6) can be satisfied for both 2D and 3D problem domains. Therefore, we have the following result.

THEOREM 5.1. *The multilevel preconditioners $M^{(k)}$ defined by* (2.7)–(2.10) *and* (2.5)–(2.6) *give optimal-order methods for k_0 sufficiently large. That is to say, they are spectrally equivalent to the corresponding stiffness matrices $A^{(k)}$, and the cost of evaluating the preconditioner is $O(n_k)$, namely, proportional to the number of unknowns.*

We use the following remarks to end this section.

REMARK 5.1. For the new hybrid algebraic multilevel preconditioners with the properly scaled and shifted Chebyshev polynomial, we can estimate $\lambda^{(s)}$ starting with $s = 1$ and setting

$$\alpha^{(s)} = \begin{cases} \dfrac{1-\gamma^2}{\lambda^{(s)}}, & \text{version (i)}, \\ \dfrac{1}{\lambda^{(s)}}, & \text{version (ii)}, \end{cases}$$

$$p_\nu^{(s)}(t) = \frac{1 + T_\nu\big((1+\alpha^{(s)} - 2t)/(1-\alpha^{(s)})\big)}{1 + T_\nu\big((1+\alpha^{(s)})/(1-\alpha^{(s)})\big)};$$

the procedure continues with $s = 2, 3, \ldots, l(k_0)$. Once an unacceptable growth of the eigenvalues $\lambda^{(s)}$ takes place, the procedure can be restarted with a larger ν. Theorem 4.4 guarantees that a reasonable stabilization of the order of magnitude of the eigenvalues $\lambda^{(s)}$ can be achieved.

REMARK 5.2. k_0 ($\geqslant 1$) should be chosen in order to balance the arithmetic work for the estimation of the eigenvalues $\lambda^{(s)}$ and the work of polynomial acceleration at every global step, in other words, to ensure the inequality $\nu < \mu^{k_0 d}$.

6. CONCLUSIONS

To solve the sparse system of linear equations with symmetric positive definite coefficient matrices resulting from the discretization of many second-order elliptic boundary-value problems by the finite-element method, we propose a class of new hybrid algebraic multilevel preconditioning methods starting with reasonably constructed approximation matrices of the stiffness matrices and applying the methodology shown in [1, 3, 10–11]. These methods are shown to be superior to the existing ones in several respects, such as their generality, their computational cost, and the constraints on their approximation matrices, etc. It is further demonstrated that the preconditioners so derived are of optimal orders of complexity for two-dimensional and three-dimensional second-order self-adjoint elliptic boundary-value problem domains, and their relative condition numbers are not only independent of the regularity of the solution, but also bounded uniformly with respect to the levels and with respect to the possible jumps of the coefficients of the considered problem, provided they occur only across edges (faces in 3D) of elements from the coarsest triangulation, and provided the triangulations are generated successively by uniform refinements, starting with the coarsest triangulation, and consistent with the problem domain. Finally, we suggest adaptive implementations of our new methods, which may be more robust and practical in concrete computations.

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REFERENCES

1 O. Axelsson and P. S. Vassilevski, Algebraic multilevel preconditioning methods. I, *Numer. Math.* 56:157–177 (1989).
2 O. Axelsson and P. S. Vassilevski, Algebraic multilevel preconditioning methods. II, *SIAM J. Numer. Anal.* 27:1569–1590 (1990).
3 O. Axelsson and P. S. Vassilevski, Algebraic multilevel preconditioning methods. III, in *Domain Decomposition Methods for Partial Differential Equations* (R. Glowinski et al., Eds.), SIAM, Philadelphia, 1991, pp. 163–177.
4 O. Axelsson and I. Gustafsson, Preconditioning and two-level multigrid methods of arbitrary degree of approximation, *Math. Comp.* 40:219–242 (1983).
5 Z. Z. Bai, Parallel Iterative Methods for Large-scale Systems of Algebraic Equations, Ph.D. Thesis, Shanghai Univ. of Science and Technology, Shanghai, June 1993.
6 Z. Z. Bai, A class of hybrid algebraic multilevel preconditioning methods, *Appl. Numer. Math.* 19:389–399 (1996).
7 Z. H. Cao and Y. C. Jiang, On algebraic multilevel preconditioning methods, *Numer. Math. J. Chinese Univ.* (*English Ser.*) 1:25–37 (1993).
8 Z. H. Cao and D. J. Evans, New variants of the algebraic multilevel preconditioning methods, *Internat. J. Comput. Math.* 47:77–84 (1993).
9 P. S. Vassilevski, Nearly Optimal Iterative Methods for Solving Finite Element Elliptic Equations Based on the Multilevel Splitting of the Matrix, Rep. 1989-09, Inst. for Scientific Computation, Univ. of Wyoming, Laramie.
10 P. S. Vassilevski, Hybrid V-cycle algebraic multilevel preconditioners, *Math. Comp.* 58:489–512 (1992).
11 H. Yserentant, On the multilevel splitting of finite element spaces, *Numer. Math.* 49:379–412 (1986).
12. D. R. Wang and Z. Z. Bai, Parallel multilevel iterative methods. *Linear Algebra Appl.* 250:317–347 (1997).

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